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LARGE GROUPS AND THEIR PERIODIC QUOTIENTS

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Abstract. We first give a short group theoretic proof of the following result of Lackenby. If G is a large group, H is a finite index subgroup of G admitting an epimorphism onto a non–cyclic free group, and g is an element of H, then the quotient of G by the normal subgroup generated by g^n is large for all but finitely many $n \in \mathbb{Z}$. In the second part of this note we use similar methods to show that for every infinite sequence of primes $(p_1, p_2, \ldots)$, there exists an infinite finitely generated periodic group Q with descending normal series $Q = Q_0 \rhd Q_1 \rhd \ldots$, such that $\bigcap_i Q_i = \{1\}$ and Q_{i-1}/Q_i is either trivial or abelian of exponent p_i.

1. Introduction

Recall that a group G is *large* if some finite index subgroup of G admits a surjective homomorphism onto a non–cyclic free group. In fact, it is easy to show that if G is large, then some finite index normal subgroup of G has a free non–cyclic quotient. Given a subset S of a group G, we denote by $\langle\langle S\rangle\rangle^G$ the normal closure of S in G. In the recent paper [4], Lackenby observed that adding higher powered relations preserves the largeness of a finitely generated group. More precisely, groups Lackenby proved the following (under the additional assumption that G was finitely generated).

Theorem 1.1. *Let G be a large group, H a normal subgroup of G of finite index admitting a surjective homomorphism onto a non–cyclic free group, $g_1, \ldots, g_k$ elements of H. Then the quotient group $G/\langle\langle g_1^n, \ldots, g_k^n\rangle\rangle^G$ is large for all but finitely many $n \in \mathbb{N}$.*

As noticed in [4], this theorem has interesting applications to Dehn surgery on 3–manifolds. It is also interested from the algebraic point of view since its iterated applications allow one to construct infinite finitely generated periodic groups. The proof of Theorem 1.1 suggested in [4] essentially uses the deep theory related to property (τ) and homology growth developed by Lackenby in his recent papers (see [4] and references therein). In the first part of this note we provide a short group theoretic proof of Theorem 1.1. In the second part we apply our method to prove a modified version of Theorem 1.1 (see Lemma 3.2) and use this version to construct infinite finitely generated periodic groups with certain specific properties.

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More precisely, let $\Pi = (p_i)$ be a (finite or infinite) sequence of primes. In this paper we say that a group Q is a Π–*graded group* if Q admits a normal series

$$Q = Q_0 \rhd Q_1 \rhd \ldots, \tag{1}$$

such that $\bigcap\limits_i Q_i = \{1\}$ and Q_{i-1}/Q_i is either trivial or abelian of exponent p_i. Note that all subgroups Q_i have finite index in Q whenever Q is finitely generated.

Theorem 1.2. *For any infinite sequence of primes $\Pi = (p_i)$, there exists a finitely generated infinite periodic Π–group.*

Observe that every finitely generated Π–group is residually finite and if Q is a periodic Π–group, then every element $x \in Q$ has order $p_{j_1} \cdots p_{j_k}$, where $p_{j_1}, \ldots, p_{j_m}$ is a subsequence of Π depending on x (see Lemma 3.3). For instance, if $\Pi = (p, p, \ldots)$, we obtain an infinite finitely generated periodic residually finite p–group. Such groups were first constructed by Golod [2] and then by Aleshin, Grigorchuk, Gupta - Sidki, Sushchansky, and others. Note that all these groups are residually nilpotent.

On the other hand, Theorem 1.2 can be applied to find examples of a quite different nature. Recall that S is a *section* of a group Q if S is a quotient group of a subgroup of Q. Recall also that an infinite group is called *just infinite* if all its proper quotients are finite.

Corollary 1.3. *There exists a finitely generated periodic just infinite group Q such that:*

1. *Orders of elements of Q are square free.*
2. *Every section of Q is residually finite. In particular, Q is residually finite.*
3. *If S is a finite section of Q, then S is solvable and all Sylow subgroups of S are abelian. In particular, every nilpotent section of Q is abelian.*

We note that the first examples of finitely generated residually finite just infinite periodic groups were constructed in [3]. However, groups from [3] were p–groups. In particular they were residually nilpotent as well as Golod groups [2] (compare with (c)). To the best of our knowledge no examples of finitely generated residually finite periodic groups satisfying either of the properties (a)–(c) were known until now.

2. Proof of Theorem 1.1

Throughout this paper F denotes a non–abelian free group. We write $H \twoheadrightarrow F$ if a group H admits an epimorphism onto some F. The main ingredient of our proof is the following.

Theorem 2.1 (Baumslag, Pride, [1])**.** *Suppose that a group G admits a presentation with $n \geq 2$ generators and at most $n-2$ relations. Then G is large.*

Remark 2.2. In fact, Baumslag and Pride proved even a stronger result. Under the assumptions of Theorem 2.1, they showed that for every sufficiently large $m \in \mathbb{N}$ there is a normal subgroup $H = H(m) \leq G$ such that $G/H \cong \mathbb{Z}/m\mathbb{Z}$ and $H \twoheadrightarrow F$ (see [1]).

Given elements g, t of a certain group, we use the notation g^t to express $t^{-1}gt$ and $C_G(g)$ to denote the centralizer of g in G. The proof of Theorem 1.1 is based on the following auxiliary results.

Lemma 2.3. *Let G be a group, N a normal subgroup of G, $g \in N$. Let T denote the set of representatives of the right cosets of CN in G, where C is an arbitrary subgroup of $C_G(g)$. Then $\langle\langle g\rangle\rangle^G = \langle\langle Z\rangle\rangle^N$, where $Z = \{g^t \mid t \in T\}$.*

Proof. It suffices to show that $g^s \in \langle\langle Z\rangle\rangle^N$ for every $s \in G$. To this end we note that $s = fht$ for some $f \in C \leq C_G(g)$, $h \in N$, and $t \in T$. Hence

$$g^s = g^{fht} = g^{ht} = g^{th'},$$

where $h' = h^t \in N$. Thus $g^s \in \langle\langle Z\rangle\rangle^N$. □

Lemma 2.4. *For any finite collection of non–trivial elements $g_1, \ldots, g_k$ of a free group F and any number $m \in \mathbb{N}$, there exists $M \in \mathbb{N}$ with the following property. For every $q \geq M$, there is a finite index normal subgroup $N \lhd F$ such that for all $1 \leq i \leq k$, $g_i^s \notin N$ whenever $1 \leq s \leq m$, but $g_i^q \in N$.*

Proof. It suffices to prove the lemma in the case when F is finitely generated. Let $a_1, ..., a_r$ be a basis of F, $A(\mathbb{Z}, r)$ (respectively $A(\mathbb{F}_p, r)$) the algebra of formal power series in non–commutative variables $x_1, \ldots, x_r$ over $\mathbb{Z}$ (respectively $\mathbb{F}_p$), X (respectively X_p) the ideal of $A(\mathbb{Z}, r)$ (respectively $A(\mathbb{F}_p, r)$) generated by $x_1, ..., x_r$. It is well known that the subset $1 + X$ of $A(\mathbb{Z}, r)$ forms a group with respect to multiplication, and the map $a_i \to 1 + x_i$, $1 \leq i \leq r$, extends to an embedding $F \to 1 + X$ [5].

Clearly there are $l, M_0 \in \mathbb{N}$ such that for any prime $p \geq M_0$ the natural image of the set $S = \{g_i^s \mid 1 \leq i \leq k, 1 \leq s \leq m\}$ in $A(\mathbb{F}_p, r)/X_p^l$ does not contain 1. Without loss of generality we may assume $M_0 \geq l$. Recall that F is residually finite p–group for any prime p. Hence for each prime $2 \leq p \leq M_0$, there is a subgroup N_p of index $p^{j(p)}$ in F for a certain $j(p) \in \mathbb{N}$ such that $S \cap N = \emptyset$. Set $M = \prod_{2 \leq p \leq M_0} p^{j(p)}$. Each number $q \geq M$ is divisible by either $p^{j(p)}$ for some prime $2 \leq p \leq M_0$, or by a prime $p > M_0 \geq l$. In the first case, we set $N = N_p$. In the second case the inequality $p > l$ implies that the image of every element of $1 + X$ in $A(\mathbb{F}_p, r)/X_p^l$ has order p. Therefore, we can we set N to be the kernel of the homomorphism of F onto its image in $A(\mathbb{F}_p, r)/X_p^l$. □

Lemma 2.5. *Let F be a free group of rank $r \geq 2$, $g_1, \ldots g_k$ arbitrary elements of F. Then $\overline{F} = F/\langle\langle g_1^q, \ldots, g_k^q\rangle\rangle^F$ is large for all but finitely many $q \in \mathbb{N}$.*

Proof. Without loss of generality we may assume that $g_1, \ldots, g_k$ are nontrivial. By Lemma 2.4, there exists $M \in \mathbb{N}$ such that for any $q \geq M$ there is a finite index normal subgroup $N \lhd F$ such that for all $i = 1, \ldots, k$, $g_i^s \notin N$ if $s = 1, \ldots, k$, but $g_i^q \in N$. Note that the image of g_i in F/N has order at least $k+1$. Therefore the index of $\langle g_i\rangle N$ in F is at most $j/(k+1)$, where $j = |G : N|$.

According to Lemma 2.3, the image $\overline{N}$ of the subgroup N in $\overline{F}$ is isomorphic to $N/\langle\langle Z\rangle\rangle^N$, where $Z = \bigcup\limits_{i=1}^{k} \{(g_i^q)^t \mid t \in T_i\}$ and T_i is the set of representatives of the right cosets of $\langle g_i\rangle N$ in F. Therefore, $\overline{N}$ admits a presentation with

$$rank\, N = 1 + (r-1)j \geq 1 + j$$

generators and

$$\#Z = \sum_{i=1}^{k} \#T_i = \sum_{i=1}^{k} |F : \langle g_i\rangle N| \leq \frac{kj}{k+1} < j$$

relations. Hence $\overline{N}$ is large by Theorem 2.1. As R is of finite index in $\overline{F}$, the group $\overline{F}$ is also large. $\square$

Proof of Theorem 1.1. Let H be a normal subgroup of finite index in G such that admitting a homomorphism $\varepsilon\colon H \to F$ onto a non–cyclic free group. By Lemma 2.3, for every $n \in \mathbb{N}$, the image $\overline{H}$ of H in $G/\langle\langle g_1^n, \ldots, g_k^n\rangle\rangle^G$ is isomorphic to $H/\langle\langle Z^n\rangle\rangle^H$, where Z is some finite subset of H consisting of conjugates of g_i's and $Z^n = \{z^n | z \in Z\}$. Thus $\overline{H}$ admits a surjective homomorphism onto $F/\langle\langle \varepsilon(Z^n)\rangle\rangle^F = F/\langle\langle \varepsilon(Z)^n\rangle\rangle^F$, which is large for all but finitely many n according to Lemma 2.5 applied to the set $\varepsilon(Z)$. Hence so are $\overline{H}$ and $G/\langle\langle g^n\rangle\rangle^G$. $\square$

3. Constructing periodic Π–graded groups

To each group G and each sequence of primes $\Sigma = (q_i)$, we associate a sequence of characteristic subgroups $\delta_i^{\Sigma}(G)$ of G defined by $\delta_0^{\Sigma}(G) = G$ and

$$\delta_i^{\Sigma}(G) = [\delta_{i-1}^{\Sigma}(G), \delta_{i-1}^{\Sigma}(G)] \left(\delta_{i-1}^{\Sigma}(G)\right)^{q_i}.$$

It is easy to prove by induction that these subgroups have finite index in G whenever G is finitely generated. We need an auxiliary lemma, which follows immediately from a result of Levi (see [6, Lemma 21.61]).

Lemma 3.1. *For any finite subset S of nontrivial elements of a free group F, there exists $D \in \mathbb{N}$ such that for any infinite sequence of primes Σ, we have $\delta_d^{\Sigma}(F) \cap S = \emptyset$ for all $d \geq D$.*

Throughout the rest of this section we fix an arbitrary infinite sequence $\Pi = (p_i)$ of primes and denote by ω_k the subsequence $(p_{k+1}, p_{k+2}, \ldots)$ of Π.

Lemma 3.2. *Let G be a finitely generated group. Suppose that $\delta_r^{\Pi}(G) \twoheadrightarrow F$ for some r. Then for any element $g \in \delta_r^{\Pi}(G)$ there is $m \geq r$ such that if $g^n \in \delta_m^{\Pi}(G)$, then $\delta_m^{\Pi}(G)/\langle\langle g^n\rangle\rangle^G \twoheadrightarrow F$.*

Proof. By Lemma 2.3, $\delta_r^{\Pi}(G)/\langle\langle g^n\rangle\rangle^G$ is isomorphic to $\delta_r^{\Pi}(G)/\langle\langle h_1^n, \ldots, h_k^n\rangle\rangle^{\delta_r^{\Pi}(G)}$, where $h_1, \ldots, h_k$ are some conjugates of g. Thus $\delta_r^{\Pi}(G)/\langle\langle g^n\rangle\rangle^G$ surjects onto $\overline{F} = F/\langle\langle g_1^n, \ldots, g_k^n\rangle\rangle^F$, where g_i stands for the image of h_i in F. Without loss of generality we may assume that $g_1, \ldots, g_k$ are non–trivial. By Lemma 3.1, there exists d such that $g_i^s \notin \delta_d^{\omega_r}(F)$ for $1 \leq i, s \leq k$. We denote the subgroup $\delta_d^{\omega_r}(F)$ by N.

Assume that $g^n \in \delta_{r+d}^{\Pi}(G)$. Then $g_i^n \in N$ for all i. We now repeat word–for–word the arguments from the proof of Lemma 2.5 and conclude that the image $\overline{N}$ of N in $\overline{F}$ admits a presentation with 2 more generators than relations. According to Remark 2.2, there is a subgroup $M \lhd \overline{N}$ such that $\overline{N}/M$ is cyclic of order $p_{r+d+1} \cdots p_{r+d+c}$ for some $c \geq 1$ and $M \twoheadrightarrow F$. Clearly $\delta_{c+d}^{\omega_r}(\overline{N})$ is a finite index subgroup of M and hence $\delta_{c+d}^{\omega_r}(\overline{N}) \twoheadrightarrow F$. It remains to note that $\delta_{r+d+c}^{\Pi}(G)/\langle\langle g^n\rangle\rangle^G$ surjects onto $\delta_{c+d}^{\omega_r}(\overline{N})$ and set $m = r + d + c$. $\square$

Proof of Theorem 1.2. We enumerate all elements of the free group $F = \{f_1, f_2, \ldots\}$ and construct a sequence of quotients G_i of F as follows. Let $G_0 = F$ and suppose that we have already constructed G_i such that

$$\delta_{r_i}^{\Pi}(G_i) \twoheadrightarrow F \tag{2}$$

for some $r_i \in \mathbb{N}$. Let also g_{i+1} denote the image of $(f_{i+1})^{p_1 p_2 \cdots p_{r_i}}$ in G_i. Then $g_{i+1} \in \delta^{\Pi}_{r_i}(G)$ and we can choose $r_{i+1} > r_i$ and $n = n(i) \in \mathbb{N}$ according to Lemma 3.2 so that $g^n_{i+1} \in \delta^{\Pi}_{r_{i+1}}(G_i)$ and $\delta^{\Pi}_{r_{i+1}}(G_i)/\langle\langle g^n_{i+1}\rangle\rangle^{G_i} \twoheadrightarrow F$. We set $G_{i+1} = G_i/\langle\langle g^n_{i+1}\rangle\rangle^{G_i}$. Clearly $\delta^{\Pi}_{r_{i+1}}(G_{i+1}) \twoheadrightarrow F$. Observe also that (2) and inequality $r_{i+1} > r_i$ imply that $\delta^{\Pi}_{r_{i+1}}(G_i)$ is a proper subgroup of $\delta^{\Pi}_{r_i}(G_i)$. Therefore,

$$|G_{i+1}/\delta^{\Pi}_{r_{i+1}}(G_{i+1})| = |G_i/\delta^{\Pi}_{r_{i+1}}(G_i)| > |G_i/\delta^{\Pi}_{r_i}(G_i)|. \tag{3}$$

Let K_i denote the kernel of the natural homomorphism $F \to G_i$. Clearly $G = F/\bigcup\limits_{i=1}^{\infty} K_i$ is a periodic group. Further set $Q = G/\bigcap\limits_{i=1}^{\infty} \delta^{\Pi}_{r_i}(G)$. Obviously Q is a Π–group. To show that Q is infinite we observe that $Ker(G_i \to G) \leq \delta^{\Pi}_{r_i}(G_i)$ for every i. Hence

$$G/\delta^{\Pi}_{r_i}(G) \cong G_i/\delta^{\Pi}_{r_i}(G_i).$$

Inequality (3) now implies $|G/\delta^{\Pi}_{r_i}(G)| \to \infty$ as $i \to \infty$. Therefore, the group Q is infinite as it surjects onto $G/\delta^{\Pi}_{r_i}(G)$ for every i. □

To derive Corollary 1.3 we first list some general properties of Π–graded groups in the case when Π consists of distinct primes.

Lemma 3.3. *Suppose that all primes in the sequence $\Pi = (p_1, p_2, \ldots)$ are distinct and Q is a periodic Π–graded group with normal series (1). Then the following holds.*

- (a) *Every element $x \in Q$ has order $p_{j_1} \ldots p_{j_k}$, where $p_{j_1}, \ldots, p_{j_k}$ is a subsequence of Π. Moreover, if $x \in Q_i$, then $j_1 > i$.*
- (b) *Every subgroup of Q is a Π–graded group.*
- (c) *Every quotient $\overline{Q}$ of Q is a Π–graded group.*
- (d) *If Q is finitely generated, then every section of Q is residually finite.*
- (e) *Every finite section of Q is solvable. Every nilpotent section of Q is abelian.*

Proof. To prove (a) we note that for every element $x \in Q$, there exists m such that $Q_m \cap \langle x \rangle = \{1\}$. Hence the order of x in Q coincides with the order of the image of x in the Π–graded group Q/Q_m. Thus it suffices to prove (a) for Π–graded groups admitting finite series of type (1). This can easily be done by induction on the length of the series.

Assertion (b) is trivial. To prove (c) it suffices to note that the intersection I of the images of subgroups Q_i in $\overline{Q}$ is periodic, but I can not have elements of any prime order according to (a).

If R is a subgroup of Q and $\overline{R}$ is a quotient of R, then the same argument as above shows that the intersection of images of $\overline{R}_i = R \cap Q_i$ in $\overline{R}$ is trivial. Since Q is finitely generated, $\overline{R}/\overline{R}_i \cong RQ_i/Q_i \leq Q/Q_i$ is finite. This proves (c).

Note that every section of Q is a Π–graded group by (b) and (c). Every finite Π–graded group admits a series (1) of finite length, hence every such a group is solvable. Finally we recall that periodic nilpotent groups are locally finite and finite nilpotent groups are direct products of their Sylow subgroups. Obviously every finite Π–graded p–group is abelian whenever Π consists of distinct primes. This implies (d). □

Proof of Corollary 1.3. Suppose that Π consists of distinct primes and Q is a Π–graded group provided by Theorem 1.2. As Q is finitely generated and infinite, by

the Zorn Lemma there is a just infinite quotient $\overline{Q}$ of Q. The group $\overline{Q}$ is also a Π–group and have all required properties by Lemma 3.3. □

Finally we mention one question motivated by Theorem 1.2 and Corollary 1.3.

Problem 3.4. *Does there exist an infinite finitely generated periodic residually (finite simple) group?*

Another interesting question is whether there exists an infinite finitely generated residually finite group that is also residually simple. Problem 3.4 is related to one of the main open questions in the theory of hyperbolic groups, that is, whether every hyperbolic groups is residually finite. If this question has positive answer, our method can be used to construct finitely generated infinite periodic residually (finite simple) groups. Here is the sketch of the proof.

In [8], the first author observed that if all hyperbolic groups are residually finite, then every non–elementary hyperbolic group has infinitely many finite simple quotients. Recall also that if G is a non–elementary hyperbolic group and $g \in G$, then $G/\langle\langle g^n\rangle\rangle^G$ is also non–elementary hyperbolic provided n is big enough [7]. Starting with a finitely generated non–abelian free group $F = \{f_1, f_2, \ldots\}$ and assuming that every hyperbolic group is residually finite, we can construct a sequence of non–elementary hyperbolic quotients G_i of F and subgroups $N_i \lhd G_i$ such that $G_{i+1} = G_i/\langle\langle g_{i+1}^n\rangle\rangle^{G_i}$ for some $n = n(i)$, G_i/N_i is finite simple, $|G_i/N_i| \to \infty$ when $i \to \infty$, and for every i, g_{i+1}^n belongs to the intersection of the images of all subgroups $N_0, N_1, \ldots, N_i$ in G_i. Here g_i is the image of f_i in G_i as above. Then we define the quotient group G of F as in the proof of Theorem 1.2 and denote by Q the quotient of G by the intersection of images of all subgroups N_i, $i \in \mathbb{N}\cup\{0\}$ in G. Clearly Q is finitely generated, periodic, residually (finite simple), and infinite for the same reason as above.

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